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SMALL PERTURBATION IN THE PROBLEM OF JET IMPINGEMENT: CONSTITUTIVE EQUATIONS

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The well-known steady-state problem of impingement of two plane jets of an ideal incompressible fluid moving with the same velocity is refined and extended to the case of unsteady interaction. Equations describing perturbation propagation on the free surface of the impinging jets are obtained and linearized on the steady-state solution.

Key words: *ideal incompressible fluid, free surface, unsteady jet flows, linearization.*

Introduction. The problem of impingement of two jets moving at an angle toward each other with the same velocity which can be considered equal to unity without loss of generality is one of the simplest problems solved by the Kirchhoff method [1]. It arises from the problem of fluid flow from two hoses (more precisely, slots) of different thicknesses (Fig. 1a) in the limit where these hoses move to infinity in directions opposite to the directions of the jet velocities. This results in a steady-state jet flow which is shown schematically in Fig. 1b.

Investigation of unsteady flow is necessary, for example, for impingement of jets moving at different velocities. In this case, a continuous steady-state solution cannot be constructed because different Bernoulli constants cannot provide pressure continuity at the critical point. A moving coordinate system in which the flow is steady-state can be found only in the particular case of head-on impingement where the jet velocities are different and directed toward each. In the case where the jets impinge at an angle, such a system does not exist. Unlike in the case of head-on impingement, in this case there is a distinguished point, for example, the point of intersection of the midline of the impingement jets (point A in Fig. 1a). In a coordinate system rigidly attached to this point, the flow is apparently periodic in time, which is confirmed by experimental studies [2] which showed the existence of aperiodic waves on the free surface during interaction of two oil flows moving at different velocities. This periodic interaction of jets can be regarded as the simplest model of wave formation during explosion welding [3]. More complex models taking into account acoustic perturbations, viscosity, thermal conductivity, surface tension, instability of the von Kármán vortex street, etc., are presented in [4].

The equations obtained below can be used to describe the interaction of jets moving at different velocities. However, first of all, it is necessary to derive equations to study the stability of steady-state impingement of jets moving at the same velocity. In this case, it is assumed that perturbations decrease at infinity, i.e., do not influence the steady-state flow velocity at points at infinity.

Slightly perturbed jet flows are commonly studied by using the equations given in [5] with the boundary conditions extended to the unperturbed free surface. The question of the validity of this procedure for the case of unsteady interaction of jets where there are four points at infinity remains open. In this case, in fact, a conformal mapping of the perturbed flow region onto the unperturbed region is performed. It is not clear whether it is possible to reach a correspondence of the points at infinity because conformal mapping is known to be only a three-parameter one. To avoid errors, we first rigorously formulate the nonlinear problem, separately stating the assumptions used for this, and only then we perform linearization of the problem.

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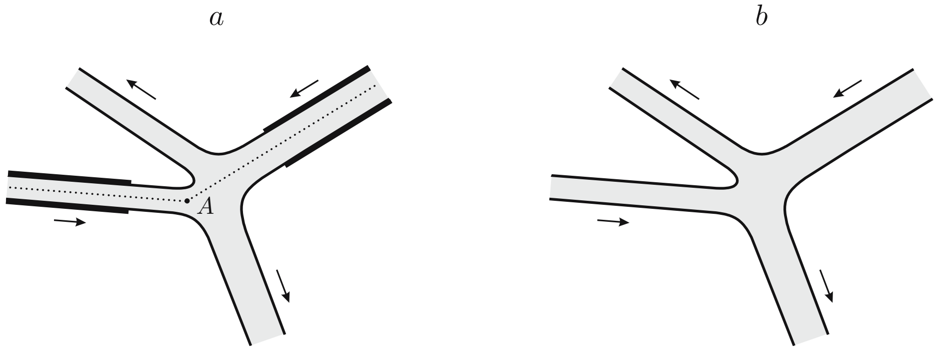

Fig. 1. Diagram of steady-state impingement of jets: (a) flow from two slots; (b) flow from infinity; arrows show the jet velocity direction.

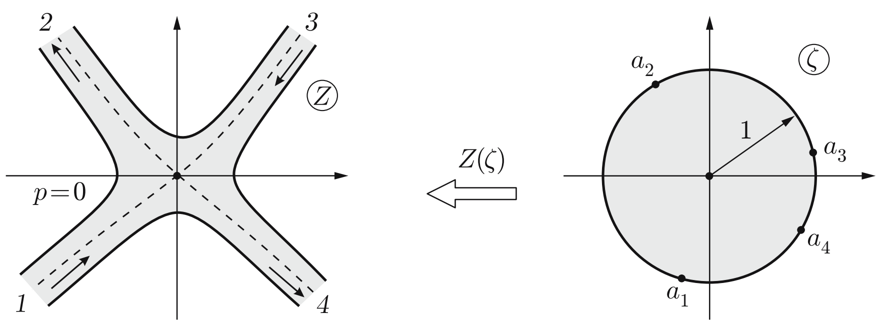

Fig. 2. Conformal mapping $Z(\zeta)$ of a circle of unit radius in the auxiliary plane ζ onto the flow region in the plane Z in the case of steady-state interaction of jets: convergent jets (1 and 3) and divergent jets (2 and 4); arrows show the jet velocity directions.

Steady-State Solution. The interaction of two convergent jets results in the formation of two divergent jets (Fig. 2). We use h_j to denote the width of the jth jet ($j = \overline{1,4}$). In the case of convergent jets, $h_1 > 0$ and $h_3 > 0$, and in the case of divergent jets, $h_2 < 0$ and $h_4 < 0$. Let $a_j = \mathrm{e}^{i\theta_j}$ be complex velocities at points at infinity. The argument θ_j characterizes the angle at which the jth jet is directed. We place the origin of Cartesian coordinates X, Y ($Z = X + iY$) at the critical point, i.e., at the point at which the fluid velocity is equal to zero. Conformal mapping of the unit circle located in a certain auxiliary plane ζ onto the flow region (see Fig. 2) and the complex velocity U expressed in terms of ζ are given by the formulas [1]

$$Z(\zeta) = \frac{1}{\pi} \sum_{j=1}^{4} \frac{h_j}{a_j} \ln\Big(1 - \frac{\zeta}{a_j}\Big), \qquad U(\zeta) = \zeta. \tag{1}$$

The points a_j located on the unit circle $|\zeta| = 1$ are preimages of the infinitely remote points of the jets in the plane Z. The second equation in (1) implies that the plane ζ is the hodograph plane.

Notwithstanding that the parameters of the impinging jets are given, the parameters the of divergent jets cannot be determined uniquely. The parameters of the convergent jets are given by the quantities h_1, h_3, a_1, and a_3. To find the four parameters of the divergent jets h_2, h_4, a_2, and a_4, we have only three equations:

$$\sum_{j=1}^{4} h_j = 0; \tag{2}$$

$$\sum_{j=1}^{4} h_j a_j = 0; \tag{3}$$

$$\sum_{j=1}^{4} \frac{h_j}{a_j} = 0. \tag{4}$$

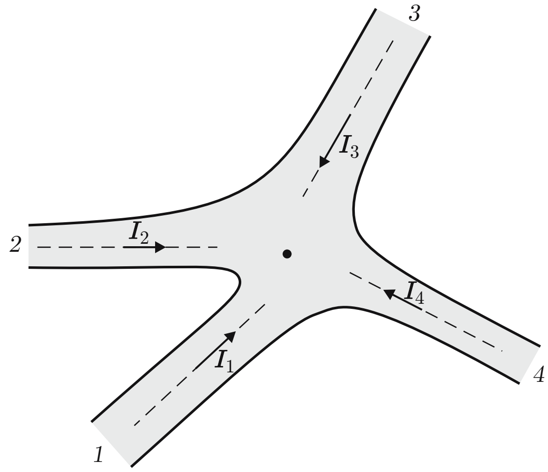

Fig. 3. Geometrical interpretation of the momentum conservation law (notation the same as in Fig. 2).

Equation (2) is the mass conservation law, and Eqs. (3) and (4) are obtained from the momentum conservation law. The energy conservation law implies an expression coincident with (2). The momentum conservation law admits the following geometrical interpretation. If we introduce auxiliary vectors $\boldsymbol{I}_k$, each of which is directed along the midline of the kth jet to the critical point and has length equal to the thickness of the kth jet (Fig. 3), the sum of momenta of the four vectors $\boldsymbol{I}_k$ relative to any point will be equal to zero. It is easy to show that in this formulation, the momentum conservation law holds identically for any flow (1).

Thus, the solution of the jet impingement problem is not unique. We specify one parameter of the divergent jets, for example, the angle θ_2 (or a_2); then, from Eqs. (2)–(4), we find the remaining parameters:

$$a_4 = \frac{a_1 a_3[h_1(a_1 - a_2) + h_3(a_3 - a_2)]}{h_1 a_3(a_1 - a_2) + h_3 a_1(a_3 - a_2)}, \qquad h_2 = -\frac{h_1 h_3 a_2 (a_1 - a_3)^2}{h_1 a_3 (a_1 - a_2)^2 + h_3 a_1 (a_3 - a_2)^2},$$

$$h_4 = -h_1 - h_3 - h_2.$$

Substitution of $a_j = \mathrm{e}^{i\theta_j}$ yields

$$h_2 = \frac{h_1 h_3 [1 - \cos(\theta_1 - \theta_3)]}{h_1[\cos(\theta_1 - \theta_2) - 1] + h_3[\cos(\theta_3 - \theta_2) - 1]}.$$

Fixing θ_2, we obtain flow which will be called the jet configuration. Changing the value of θ_2 in a certain range, we find a one-parameter family of jet configurations for each of which the parameters of the impinging jets (thickness and direction) are fixed. The streamlines of some possible jet configurations are shown in Figs. 4 and 5.

Previously, attempts have been made to find a unique solution of the steady-state problem of jet impingement by obtaining the fourth closing equation which makes the problem determined. Palatini [6] was the first to calculate the kinetic energy of the fluid in the circle $|\zeta| < R$ $(0 < R < 1)$ and showed that, for all values of R, the minimum of this energy is reached if the directions of the divergent jets are opposite. This gives the closing equation $\theta_4 = \theta_2 + \pi$. In later studies [7–9], the fourth closing equation was found by various methods and, as a result, the single flows obtained in these studies were different.

Apparently, the more correct point of view is the one expressed by Milne-Thomson, according to which the uncertainty is undoubtedly due to the fact that we consider steady-state motion that has already developed and ignore the initial conditions that led to the steady-state motion considered. According to Milne-Thomson, we can assume, for example, that the examined motion results from the start of jets at different points at different times at an interval t; there is no doubt that to different times t there correspond different steady-state motions although there is no reason to believe that all of them are steady-state [1]. In other words, each jet configuration is determined by the prehistory of its formation. Therefore, to choose the jet configuration, it is necessary to consider unsteady motion, and to determine the equivalence of jet configurations, it is reasonable to study them for stability.

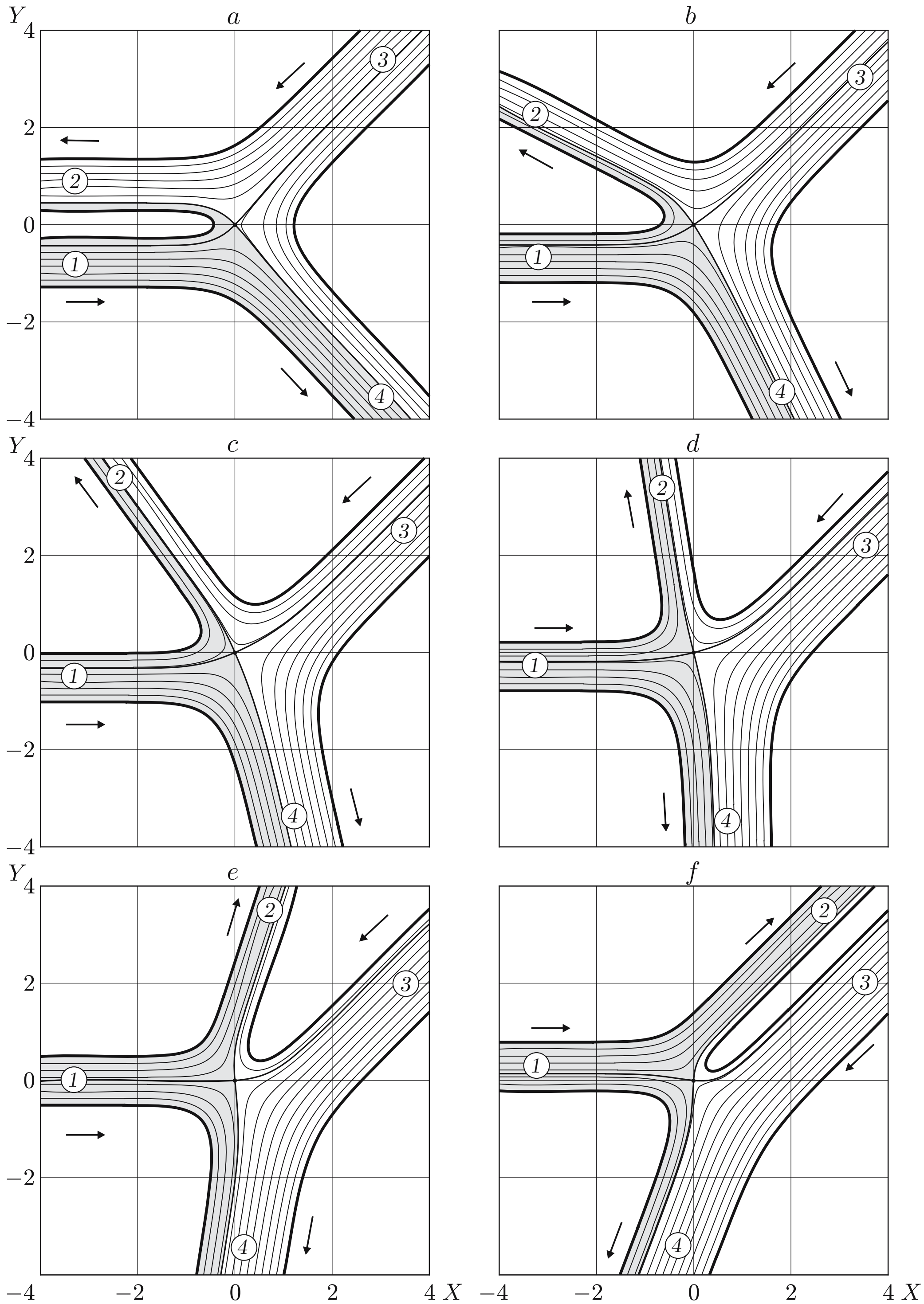

Fig. 4. Streamlines of possible jet configurations for fixed parameters of convergent jets ($\theta_1 = 0$, $\theta_3 = 3\pi/4$, $h_1 = 1$, and $h_3 = 1.5$) and various parameters of divergent jets: (a) $\theta_2 = -\pi$, $\theta_4 = 0.2611\pi$, $h_2 = -1.0497$, and $h_4 = -1.4502$; (b) $\theta_2 = -0.85\pi$, $\theta_4 = 0.3541\pi$, $h_2 = -0.8746$, and $h_4 = -1.6253$; (c) $\theta_2 = -0.7\pi$, $\theta_4 = 0.4271\pi$, $h_2 = -0.7707$, and $h_4 = -1.7292$; (d) $\theta_2 = -0.55\pi$, $\theta_4 = 0.4905\pi$, $h_2 = -0.7237$, and $h_4 = -1.7762$; (e) $\theta_2 = -0.4\pi$, $\theta_4 = 0.5513\pi$, $h_2 = -0.7259$, and $h_4 = -1.7740$; (f) $\theta_2 = -0.25\pi$, $\theta_4 = 0.6153\pi$, $h_2 = -0.7776$, and $h_4 = -1.7223$; arrows show the jet velocity directions.

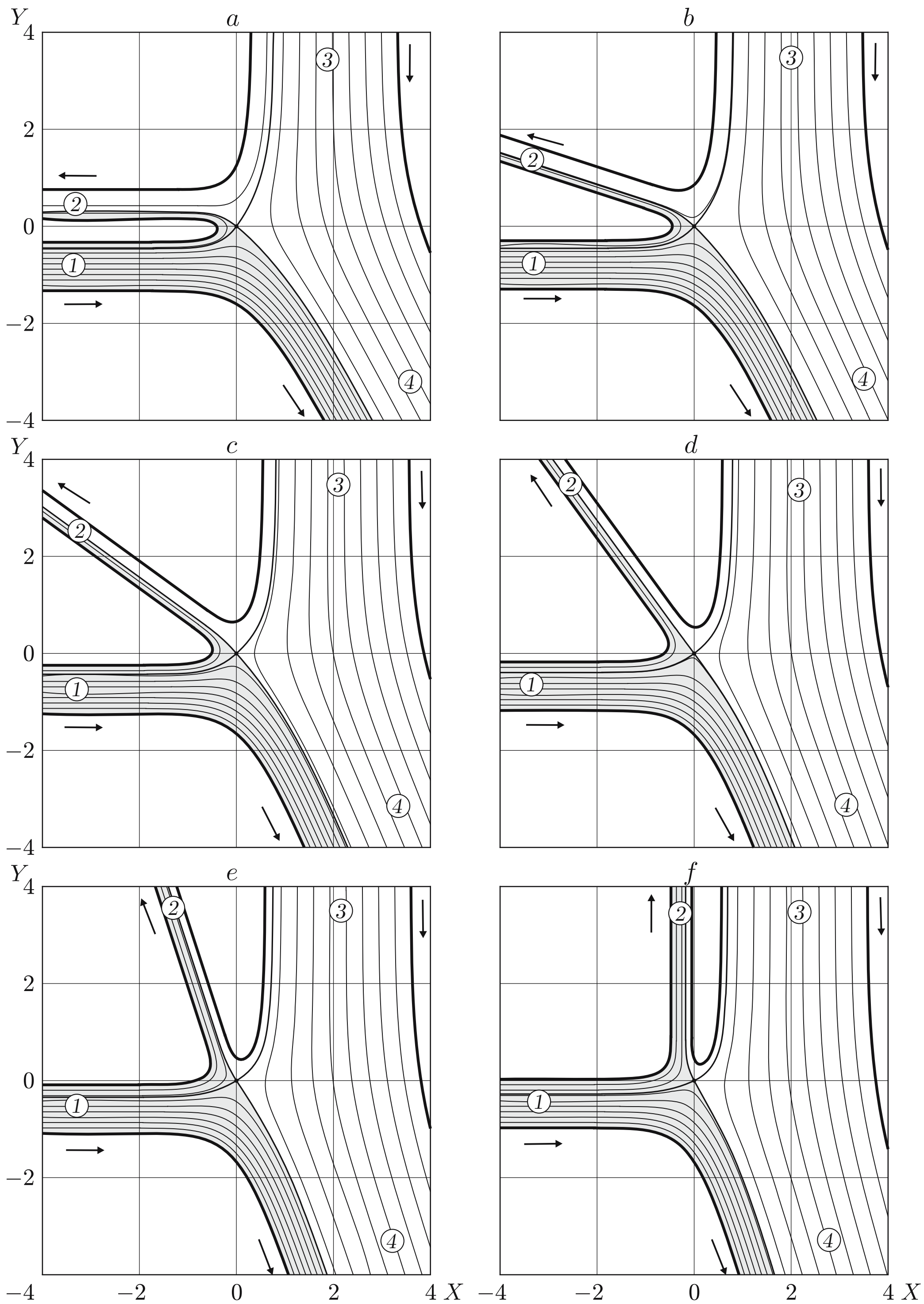

Fig. 5. Streamlines of possible jet configurations for fixed parameters of convergent jets ($\theta_1 = 0$, $\theta_3 = \pi/2$, $h_1 = 1$, and $h_3 = 3$) and various parameters of divergent jets: (a) $\theta_2 = -\pi$, $\theta_4 = 0.3440\pi$, $h_2 = -0.6$, and $h_4 = -3.4$; (b) $\theta_2 = -0.9\pi$, $\theta_4 = 0.3600\pi$, $h_2 = -0.5103$, and $h_4 = -3.4896$; (c) $\theta_2 = -0.8\pi$, $\theta_4 = 0.3737\pi$, $h_2 = -0.4564$, and $h_4 = -3.5435$; (d) $\theta_2 = -0.7\pi$, $\theta_4 = 0.3860\pi$, $h_2 = -0.4276$, and $h_4 = -3.5723$; (e) $\theta_2 = -0.6\pi$, $\theta_4 = 0.3978\pi$, $h_2 = -0.4188$, and $h_4 = -3.5811$; (f) $\theta_2 = -\pi/2$, $\theta_4 = 0.4096\pi$, $h_2 = -0.4285$, and $h_4 = -3.5714$; arrows show the jet velocity directions.

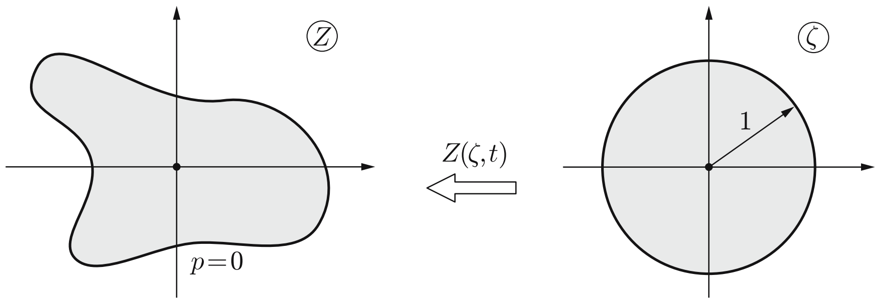

Fig. 6. Conformal mapping $Z(\zeta, t)$ of a circle of unit radius in the auxiliary plane ζ onto the variable region in the plane Z occupied by the fluid in the problem of plane unsteady flow.

We introduce the constants

$$M_n = \frac{1}{\pi}\sum_{j=1}^{4}\frac{h_j}{a_j^n} \tag{5}$$

(n is any integer) which characterizes the jet configuration considered. From (2)–(4), it follows that $M_0 = M_1 = M_{-1} = 0$.

Equations of Unsteady Fluid Motion. We consider the case of plane unsteady potential flow where the fluid occupies a bounded simply-connected region and the entire boundary of the region is free. At the initial time $t = 0$, the region is known. We assume that the field of external forces is absent; therefore, the subsequent deformation of the region occurs by inertia and is due to the given initial velocity field. The problem consists of determining the shape of the region and the velocity field at the subsequent times $t > 0$ in the case where the pressure on the boundary of the region is constant and surface tension is absent.

We consider a circle of unit radius $|\zeta| < 1$ in an auxiliary plane ζ, which in this case is not the hodograph plane. Unlike in the steady-state case (1), conformal mapping of the circle $|\zeta| < 1$ onto the region occupied by the fluid (Fig. 6) and the complex velocity are functions of time:

$$Z(\zeta, t), \quad U(\zeta, t). \tag{6}$$

It should be noted that the introduction of functions (6) gives a number of advantages compared to the traditional formulations of the problem because, instead of the boundary-value problem in an unknown region with a moving boundary in the plane ζ, we obtain a boundary-value problem with a fixed boundary — the unit circle $|\zeta| = 1$. The problem of seeking two functions which are analytic in the circle $|\zeta| < 1$ — the conformal mapping $Z(\zeta, t)$ and the complex potential $\Phi(\zeta, t)$ — was first solved in [10, 11], where two cubically nonlinear operator equations were obtained. In [12], it is noted that, if one uses the complex velocity $U(\zeta, t) = \Phi_\zeta / Z_\zeta$ instead of the complex potential, the boundary conditions and the corresponding operator equations become quadratically nonlinear. In [13], it is shown that, if instead of the pair of unknown functions (6), one uses another pair $R(\zeta, t) = 1/Z_\zeta(\zeta, t)$, $U(\zeta, t)$, the corresponding operator equations resolvable for the time derivatives do not contain division and are only cubically nonlinear. This is convenient in the numerical solution of the problem, especially when it is sought in the form of series. Examples of using this procedure to solve water wave problems are given in [14, 15].

The first kinematic boundary condition implies that the projection of the velocity $\overline{U}$ of a fluid particle located on the free boundary coincide with the projection of the velocity Z_t of the boundary onto the normal to the free boundary. In other words, the vector $Z_t - \overline{U}$ is tangential to the free surface. Because, on the free surface $\zeta = \mathrm{e}^{i\theta}$, the normal vector is equal to iZ_θ and the scalar product of the vectors Z_1 and Z_2 is given by the formula $\operatorname{Re} Z_1\overline{Z_2}$, we obtain the resulting kinematic condition in the form

$$\operatorname{Im}[Z_\theta(\overline{Z_t} - U)] = 0. \tag{7}$$

The second dynamic boundary condition can be obtained from the condition of orthogonality of the acceleration to the free surface. For convenience, we denote the time in the Eulerian and Lagrangian coordinates by τ and T, respectively. Then, the dynamic condition can be written as

$$\operatorname{Re} Z_\theta U_T = 0. \tag{8}$$

We find the acceleration U_T. Because Z and τ are independent variables, we have

$$Z_\tau = Z_t + Z_\zeta \zeta_\tau = 0.$$

Consequently,

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \zeta_\tau \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial t} - \frac{Z_t}{Z_\zeta}\frac{\partial}{\partial \zeta}.$$

Taking into account that

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial \tau} + Z_T \frac{\partial}{\partial Z} = \frac{\partial}{\partial \tau} + \overline{U}\,\frac{\partial}{\partial Z}, \qquad \frac{\partial}{\partial Z} = \frac{1}{Z_\zeta}\frac{\partial}{\partial \zeta},$$

we have the formula

$$U_T = U_t - \frac{Z_t}{Z_\zeta} U_\zeta + \frac{\overline{U}}{Z_\zeta} U_\zeta.$$

Substituting this formula into (8) and then replacing $\partial/\partial\theta$ by $i\zeta\,\partial/\partial\zeta$ in (7) and (8), we obtain two boundary conditions for the two analytic functions $Z(\zeta,t)$ and $U(\zeta,t)$:

$$\operatorname{Im}\zeta(Z_\zeta U_t - U_\zeta Z_t + U_\zeta \overline{U}\,) = 0, \qquad \operatorname{Re}(\zeta Z_\zeta U - Z_t\,\overline{Z_\zeta}/\zeta) = 0, \qquad |\zeta| = 1. \tag{9}$$

We denote by S the Schwarz operator which restores the analytic function in the unit circle from the value of its real part on the boundary of the circle. This problem is solved to within an additive imaginary constant which is chosen so that the imaginary part of the obtained function $\zeta = 0$ at the center of the circle is equal to zero. Then, the relation $f = S(g)$ implies that, first, for $|\zeta| < 1$, the function $f(\zeta)$ is analytic, second, on the boundary of the circle $|\zeta| = 1$, the condition $\operatorname{Re} f = \operatorname{Re} g$ is satisfied, and, third, at the center of the circle, $\operatorname{Im} f(0) = 0$.

Let us show that from conditions (9), it is possible to obtain the following operator equations:

$$i\zeta(Z_\zeta U_t - U_\zeta Z_t) + G(\zeta,t) = 0, \qquad G(\zeta,t) = S(i\zeta U_\zeta \overline{U}\,); \tag{10}$$

$$\frac{Z_t}{\zeta Z_\zeta} - \frac{A}{\zeta} + \overline{A}\,\zeta + iB = Q(\zeta,t), \qquad Q(\zeta,t) = S\Big(\frac{\overline{U}}{\zeta Z_\zeta}\Big). \tag{11}$$

The first equation (10) is found directly from the first boundary condition (9) because the function $\zeta(Z_\zeta U_t - U_\zeta Z_t)$ is analytic. To obtain the second equation of (11), we first divide the second boundary condition (9) by $|Z_\zeta|^2$ (this can be done because, according to the conformality condition, $|Z_\zeta| \neq 0$). As a result, we obtain the boundary condition

$$\operatorname{Re}\Big(\frac{Z_t}{\zeta Z_\zeta}\Big) = \operatorname{Re}\Big(\frac{\overline{U}}{\zeta Z_\zeta}\Big), \qquad |\zeta| = 1, \tag{12}$$

in which the function $Z_t/(\zeta Z_\zeta)$ is not analytic everywhere because for $\zeta = 0$ it has a pole $A(t)/\zeta$, where

$$A(t) = \lim_{\zeta\to 0}\frac{Z_t}{Z_\zeta}. \tag{13}$$

If the quantity $A(t)/\zeta$ is subtracted from any expression taken on the circle $\zeta = \mathrm{e}^{i\theta}$ and then the same but complex-conjugate quantity $\overline{A(t)}\,\zeta$ is added to the result, the real part of the initial expression will not change. Therefore, the boundary condition (12) can be written as

$$\operatorname{Re}\Big(\frac{Z_t}{\zeta Z_\zeta} - \frac{A(t)}{\zeta} + \overline{A(t)}\,\zeta\Big) = \operatorname{Re}\Big(\frac{\overline{U}}{\zeta Z_\zeta}\Big),$$

where the left side is the real part of the analytic function. Hence, we obtain Eq. (11), in which the function $B(t)$ is determined from the condition $\operatorname{Im} Q(0,t) = 0$.

Equation (11) contains the complex-valued functions of time $A(t)$ and the real-valued functions $B(t)$ because the family of conformal mappings of the given region onto the other specified region is a three-parameter one. These functions are determined by the chosen normalization of the conformal mapping. We show, for example that specifying the normalization

$$Z(0,t) = \text{const}, \qquad \arg Z_\zeta(0,t) = \text{const}, \tag{14}$$

we have $A = 0$ and $B = 0$.

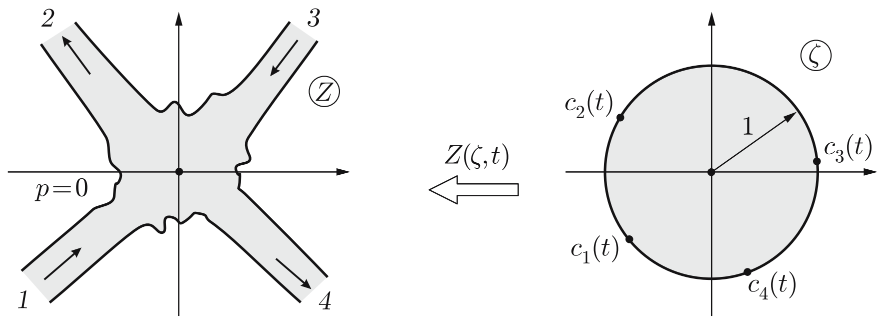

Fig. 7. Conformal mapping $Z(\zeta, t)$ of a circle of unit radius in the auxiliary plane ζ onto the flow region in the plane Z in the problem of unsteady impingement of jets (notation the same as in Fig. 2).

Indeed, because the function $Z(\zeta, t)$ is analytic at the center of the circle $\zeta = 0$, in view of the first equation of (14), this function can be expanded in the power series

$$Z(\zeta, t) = \mathrm{const} + Z_1(t)\zeta + Z_2(t)\zeta^2 + \ldots .$$

Differentiating this series with respect to time t and ζ, for $\zeta \to 0$ we have

$$Z_t/Z_\zeta = \zeta \dot{Z}_1/Z_1 + O(\zeta^2). \tag{15}$$

Substitution of expression (15) into (13) yields the equality $A = 0$, which, in view of (11), implies that $B = \lim\limits_{\zeta\to 0} \operatorname{Im} Z_t/(\zeta Z_\zeta)$. Again substituting expression (15) into the obtained relation, we have $B = \operatorname{Im} \dot{Z}_1/Z_1$. However, this quantity is equal to zero because differentiation of the second normalization condition (14) written as $\operatorname{Im} \ln Z_\zeta(0, t) = \mathrm{const}$ with respect to t gives $\operatorname{Im} Z_{\zeta t}/Z_\zeta\Big|_{\zeta=0} = 0$ or $\operatorname{Im} \dot{Z}_1/Z_1 = 0$.

Thus, with the normalization (14), we have the Cauchy problem

$$Z_t = \zeta Z_\zeta S\Big(\frac{\overline{U}}{\zeta Z_\zeta}\Big), \qquad Z(\zeta, 0) = \tilde{Z}(\zeta), \qquad U_t = \zeta U_\zeta S\Big(\frac{\overline{U}}{\zeta Z_\zeta}\Big) + \frac{iS(i\zeta U_\zeta \overline{U}\,)}{\zeta Z_\zeta}, \qquad U(\zeta, 0) = \tilde{U}(\zeta), \tag{16}$$

which describes the deformation of the region occupied by the fluid and bounded only by the free surface for the case where the shape of the region at the initial time is known and the initial velocity field is specified.

It should be noted that the second term in the second equation (16) does not contain singularities for $\zeta = 0$. Indeed, for $\zeta = 0$, the imaginary part $S(i\zeta U_\zeta \overline{U})$ vanishes according to the definition of the operator S, and the real part is vanishes by virtue of the average theorem:

$$\operatorname{Re} S(i\zeta U_\zeta \overline{U}\,)\Big|_{\zeta=0} = \frac{1}{2\pi}\int\limits_0^{2\pi} \operatorname{Re}\,(i\zeta U_\zeta \overline{U}\,)\, d\theta = \frac{1}{4\pi}\int\limits_0^{2\pi} \frac{\partial |U|^2}{\partial\theta}\, d\theta = 0.$$

Equations of Unsteady Interaction of Jets. We consider the case of unsteady interaction of jets where a certain steady-state jet configuration given by formulas (1) is subjected to a certain unsteady perturbation which is not considered small. We assume that, at the initial time, the perturbation is concentrated in a finite region of the plane Z (Fig. 7) and does not influence the points at infinity.

Unlike in the steady-state case, in the case considered, the points at infinity in the plane Z cannot be put in correspondence to fixed points in the plane ζ because the conformal mapping is a three-parameter one. Hence, the position of at least one of the four points in the plane ζ is a function of time. Generally, four moving points $\zeta = c_j(t)$ correspond to the points at infinity in the plane Z.

The question of whether Eqs. (10) and (11) describing the flow shown in Fig. 6 are applicable to the unsteady interaction of jets presented in Fig. 7 cannot be answered unambiguously because in Fig. 7 there are points at infinity and in Fig. 6 they are absent. This difference is important because Eqs. (10) and (11) were obtained under the assumption that the functions

$$Z_\zeta U_t - U_\zeta Z_t, \qquad Z_t/(\zeta Z_\zeta), \qquad \overline{U}/(\zeta Z_\zeta) \tag{17}$$

are continuous everywhere on the boundary $|\zeta| = 1$. It is necessary to elucidate whether this assumption is valid for the case of unsteady interaction of jets at the points $\zeta = c_j(t)$ at which the function Z is not bounded. We show that, with some assumptions on the nature of the flow, this assumption is valid and Eqs. (10) and (11) are suitable for describing the unsteady interaction of jets.

We assume that, at the initial time, the steady-state flow given by the parameters h_j and a_j $(j = \overline{1,4})$ is subjected to a perturbation which tends to zero as $|Z| \to \infty$, and consider the evolution of this perturbation in time. In a study [16] of the stability of jet flow from a slot, it is noted that the perturbation propagation has a coaxial nature: the perturbations in jets propagate at the same velocity as the jets and reach infinity only in infinite time. The following physical assumption therefore takes place.

Assumption 1. *Any perturbations in jets which are originally concentrated in a finite region Z of the plane cannot cause perturbation of points at infinity $|Z| \to \infty$ in finite time.*

Thus, we will consider unsteady jet flows whose thickness, direction, and jet velocity remain unchanged at infinity.

The constancy of the thickness and direction of the jets is provided by the conformal mapping representation

$$Z(\zeta, t) = \frac{1}{\pi} \sum_{j=1}^{4} \frac{h_j}{a_j} \ln\Big(1 - \frac{\zeta}{c_j(t)}\Big) + F(\zeta, t). \tag{18}$$

Indeed, if the function $F(\zeta, t)$ is analytic for $\zeta = c_j(t)$, then representation (18) implies that, in passing through the point $\zeta = c_j(t)$ on the circle $|\zeta| = 1$, the imaginary part of the expression $Z(\zeta, t)a_j$ undergoes a jump of the quantity h_j (this implies that the thickness of the jth jet is constant), and the real part of expression (18) for $\zeta = c_j(t)$ becomes infinity (this implies that the direction jth jet is constant).

If the function $U(\zeta, t)$ is analytic for $\zeta = c_j(t)$, the condition of constancy of the direction and velocity is written as

$$U(c_j(t), t) = a_j. \tag{19}$$

Unlike the function $U(\zeta, t)$, the limiting value of the function $F(\zeta, t)$ is not fixed:

$$F(c_j(t), t) = f_j(t). \tag{20}$$

The unknown functions $f_j(t)$ are found by solving the problem.

Thus, the following mathematical assumption is valid.

Assumption 2. *The functions $U(\zeta, t)$ and $F(\zeta, t)$ are analytic for $\zeta = c_j(t)$.*

Because the asymptotes of the free surface remain unchanged in time as $Z \to \infty$, the following four identities should hold:

$$\lim_{\zeta \to c_n} \operatorname{Im} \Big\{ \frac{1}{\pi} a_n \sum_{j \neq n} \frac{h_j}{a_j} \ln\Big(1 - \frac{\zeta}{c_j(t)}\Big) + a_n F(\zeta, t) \Big\} = \text{const} \qquad (n = \overline{1,4}). \tag{21}$$

These identities are consequences of the fact that the distance from the coordinate origin to the asymptotes does not change. Below, it is shown that identities (21) follow from the assumption of analyticity of the functions $U(\zeta, t)$ and $F(\zeta, t)$.

Let us show that if Assumption 2 is valid, the functions (17) are bounded for $\zeta = c_j(t)$. Indeed, differentiating representation (18), we obtain the derivatives

$$Z_\zeta = -\frac{1}{\pi} \sum \frac{h_j}{a_j c_j} \frac{1}{1 - \zeta/c_j} + F_\zeta, \qquad Z_t = -\frac{1}{\pi} \sum \frac{h_j}{a_j} \frac{\dot{c}_j}{c_j} + \frac{1}{\pi} \sum \frac{h_j}{a_j} \frac{\dot{c}_j}{c_j} \frac{1}{1 - \zeta/c_j} + F_t, \tag{22}$$

which have a pole at the point $\zeta = c_j$. However, the function $Z_\zeta U_t - U_\zeta Z_t$ at this point remains bounded. Indeed, using formulas (22), we find that for the function considered, the coefficient for $1/(1 - \zeta/c_j)$ is equal to $-\dfrac{1}{\pi} \dfrac{h_j}{a_j c_j} \lim\limits_{\zeta \to c_j} (U_\zeta \dot{c}_j + U_t)$. We show that this coefficient is equal to zero. Indeed, from the analyticity of the function $U(\zeta, t)$ for $\zeta = c_j$ and from formula (19) follows that this function is expanded in the series

$$U = a_j + u_1(\zeta - c_j) + u_2(\zeta - c_j)^2 + \ldots.$$

Combining the derivatives of this series with respect to t and ζ

$$U_t = \dot u_1(\zeta - c_j) + u_1(-\dot c_j) + 2u_2(\zeta - c_j)(-\dot c_j) + \ldots,$$

$$U_\zeta = u_1 + 2u_2(\zeta - c_j) + \ldots,$$

we find that, at the point $\zeta = c_j$, the quantity $U_t + U_\zeta \dot c_j$ has first-order zero:

$$U_t + U_\zeta \dot c_j = \dot u_1(\zeta - c_j) + \ldots.$$

In (17), the second and third functions $Z_t/(\zeta Z_\zeta)$ and $\overline{U}/(\zeta Z_\zeta)$ also are bounded at the point $\zeta = c_j$ because directly from formulas (22), it follows that

$$\lim_{\zeta\to c_j} \frac{Z_t}{\zeta Z_\zeta} = -\frac{\dot c_j}{c_j}, \qquad \lim_{\zeta\to c_j} \frac{\overline{U}}{\zeta Z_\zeta} = 0.$$

Thus, if the functions $F(\zeta, t)$ and $U(\zeta, t)$ are analytic for $\zeta = c_j(t)$, this class of flows can be described by system (10), (11).

Equations without Singularities. System (10), (11) is inconvenient for the analysis and numerical solution of the problem because the function $Z(\zeta, t)$ and its derivatives have singularities. Distinguishing these singularities, we obtain new equations for the six unknown functions $c_j(t)$, $F(\zeta, t)$, and $U(\zeta, t)$. Substitution of derivatives (22) into the second equation of (11) yields

$$\Big(F_t - \frac{1}{\pi}\sum \frac{h_j}{a_j}\frac{\dot c_j}{c_j} + \frac{1}{\pi}\sum \frac{h_j}{a_j}\frac{\dot c_j}{c_j}\frac{1}{1-\zeta/c_j}\Big) \Big/ \Big(\zeta F_\zeta - \frac{1}{\pi}\sum\frac{h_j}{a_j}\frac{1}{1-\zeta/c_j}\Big) = Q(\zeta, t) + \frac{A}{\zeta} - \overline{A}\,\zeta - iB. \tag{23}$$

Passing to the limit $\zeta \to c_j$ in (23), we find the first four equations for the functions $c_j(t)$:

$$\dot c_j/c_j + A/c_j - \overline{A}\,c_j - iB = -Q_j. \tag{24}$$

Here $Q_j = Q(c_j(t), t)$. Substitution of $\dot c_j$ from (24) into (23) yields the fifth equation

$$F_t = -\frac{A}{\pi}\sum\frac{h_j}{a_j c_j} - \frac{1}{\pi}\sum\frac{h_j}{a_j}\,Q_j + \zeta F_\zeta\Big(\frac{A}{\zeta} - \overline{A}\,\zeta - iB\Big) + \zeta F_\zeta Q + \frac{1}{\pi}\sum\frac{h_j}{a_j}\,\frac{Q_j - Q}{1-\zeta/c_j}. \tag{25}$$

The sixth equation is obtained from relations (10):

$$U_t = \Big\{iS(\operatorname{Re} i\zeta U_\zeta\,\overline{U}\,) + \zeta U_\zeta\Big[F_t + \frac{\zeta}{\pi}\sum\frac{h_j}{a_j c_j^2}\,\frac{\dot c_j}{1-\zeta/c_j}\Big]\Big\} \Big/ \Big(\zeta F_\zeta - \frac{1}{\pi}\sum\frac{h_j}{a_j}\,\frac{1}{1-\zeta/c_j}\Big). \tag{26}$$

In (26), it is necessary to substitute the time derivatives $\dot c_j$ and F_t from expressions (24) and (25).

Thus, if the initial conditions are given by

$$U(\zeta, 0) = \tilde U(\zeta), \qquad F(\zeta, 0) = \tilde F(\zeta), \qquad c_j(0) = \tilde{c_j}, \tag{27}$$

the evolution of the perturbations propagating in the jets is described by the Cauchy problem (24)–(27) for the system of six equations resolvable for the time derivatives $\dot c_j$, F_t, and U_t. In this case, the initial conditions $\tilde U(\zeta)$, $\tilde F(\zeta)$ should be analytic for $\zeta = \tilde{c}_j$. In addition, from (19) it follows that the function $\tilde U(\zeta)$ should be subject to the constraint $\tilde U(\tilde{c_j}) = a_j$. Additional constraints on the function $\tilde F(\zeta)$ are not required.

We note that the last term in Eq. (25) has no poles. If, in this term, we pass to the limit $\zeta \to c_j$, the multiplier $(Q_j - Q)/(c_j - \zeta)$ becomes the derivative Q_ζ. We pass to this limit another way. We first calculate the derivative of ζ with respect to relation (23):

$$\Big\{\Big(F_{\zeta t} + \frac{1}{\pi}\sum\frac{h_j \dot c_j}{a_j c_j^2}\,\frac{1}{(1-\zeta/c_j)^2}\Big)\Big(\zeta F_\zeta - \frac{1}{\pi}\sum\frac{h_j}{a_j}\,\frac{1}{1-\zeta/c_j}\Big)$$

$$-\Big[F_t - \frac{1}{\pi}\sum\frac{h_j \dot c_j}{a_j c_j}\Big(1 - \frac{1}{1-\zeta/c_j}\Big)\Big]\Big(\Big(\zeta F_\zeta\Big)_\zeta - \frac{1}{\pi}\sum\frac{h_j}{a_j c_j}\,\frac{1}{(1-\zeta/c_j)^2}\Big)\Big\}$$

$$\Big/ \Big(\zeta F_\zeta - \frac{1}{\pi}\sum\frac{h_j}{a_j}\,\frac{1}{1-\zeta/c_j}\Big)^2 + \frac{A}{\zeta^2} + \overline{A} = Q_\zeta, \tag{28}$$

and then we consider the limit $\zeta \to c_n$. The terms containing the multiplier $1/(1-\zeta/c_n)^3$ in the numerator of fraction (28) are cancelled, and the terms with the multiplier $1/(1-\zeta/c_n)^2$ therefore become higher terms. Collecting these terms in the numerator and denominator, we have

$$\lim_{\zeta\to c_n}\Big[\frac{h_n\dot c_n}{\pi a_n c_n^2}\Big(\zeta F_\zeta - \frac{1}{\pi}\sum_{j\neq n}\frac{h_j}{a_j}\,\frac{1}{1-\zeta/c_j}\Big)$$

$$+\frac{h_n}{\pi a_n c_n}\Big(F_t - \frac{1}{\pi}\sum\frac{h_j\dot c_j}{a_j c_j} + \frac{1}{\pi}\sum_{j\neq n}\frac{h_j\dot c_j}{a_j c_j}\,\frac{1}{1-\zeta/c_j}\Big)\Big]\Big/\Big(\frac{h_n^2}{\pi^2 a_n^2}\Big) + \frac{A}{c_n^2} + \overline{A} = \lim_{\zeta\to c_n} Q_\zeta. \tag{29}$$

Using the formula $\dot f_n = \lim_{\zeta\to c_n}(F_t + F_\zeta \dot c_n)$ obtained by differentiation of expressions (20) with respect to t, transforming the sums in (29) according to the rule

$$\frac{1}{\pi}\sum_{j\neq n}\frac{h_j}{a_j}\,\frac{\dot c_j/c_j - \dot c_n/c_n}{1-c_n/c_j} = \frac{1}{\pi}\sum_{j\neq n}\frac{h_j}{a_j}\,\frac{\dot c_j-\dot c_n}{c_j - c - n} - \frac{1}{\pi}\,\frac{\dot c_n}{c_n}\sum_{j\neq n}\frac{h_j}{a_j} \tag{30}$$

and taking into account that, according to (4), the last sum of expression (30) is equal to $-h_n/a_n$, we obtain

$$\frac{d}{dt}\lim_{\zeta\to c_n(t)}\Big[\frac{1}{\pi}\,a_n\sum_{j\neq n}\frac{h_j}{a_j}\ln\Big(1-\frac{\zeta}{c_j(t)}\Big) + a_n F(\zeta,t)\Big] = -\frac{h_n}{\pi}\Big(\frac{A}{c_n} + \overline{A}\,c_n\Big) + \frac{h_n}{\pi}\,c_n Q_\zeta(c_n,t). \tag{31}$$

Using the imaginary part in (31), we obtain identity (21) because the imaginary part of the first term on the right side of (31) is equal to zero, and the vanishing of the imaginary part of the second term follows from the definition of the function Q and the equalities

$$\operatorname{Im} c_n Q_\zeta(c_n,t) = \lim_{\zeta\to c_n}\operatorname{Im}\zeta Q_\zeta = -\lim_{\zeta\to c_n}\operatorname{Re} Q_\theta = -\lim_{\zeta\to c_n}\operatorname{Re}\Big(\frac{\overline{U}}{\zeta Z_\zeta}\Big)_\theta = \lim_{\zeta\to c_n}\operatorname{Im} i\zeta\Big(\frac{\overline{U}}{\zeta Z_\zeta}\Big)_\zeta = 0.$$

The last limit vanishes because the function U is bounded and by virtue of representation (18).

From the form of operator system (24)–(26), it follows that if the functions $F(\zeta,t)$ and $U(\zeta,t)$ are analytic at the points $\zeta = c_j$, the time derivatives $F_t(\zeta,t)$ and $U_t(\zeta,t)$ are also analytic at these points. This suggests that the examined class of flows exists. In other words, if the initial conditions (27) are analytic at $\zeta = c_j$, the functions $F(\zeta,t)$ and $U(\zeta,t)$ have this property at the subsequent times as well.

It should be noted that if the analytic functions are considered at $\zeta = c_j$ the problem is simplified and the class of solutions is narrowed. However, it is possible to weaken assumption 2 and replace analyticity by boundedness. In other words, under the assumption that, at $\zeta = c_j$, the functions $F(\zeta,t)$ and $U(\zeta,t)$ and their first derivatives are bounded, Eqs. (10) and (11) are still applicable to this class of solutions. Moreover, from (21), it follows that only the imaginary part of the expression $a_j F(\zeta,t)$ should be limited and the real part can increase without bound. If this increase is not too rapid, the perturbations at infinity still decrease but at a lower velocity than in the case of analyticity or boundedness, and Eqs. (10) and (11) are also applicable in this case.

Linearization. We find a solution of system (24)–(26) that differs only slightly from the steady-state solution (1):

$$U(\zeta,t;\varepsilon) = U^{(0)}(\zeta) + \varepsilon U^{(1)}(\zeta,t) + \ldots,\qquad F(\zeta,t;\varepsilon) = F^{(0)}(\zeta) + \varepsilon F^{(1)}(\zeta,t) + \ldots,$$
$$c_j(t;\varepsilon) = c_j^{(0)} + \varepsilon c_j^{(1)}(t) + \ldots\qquad (j = \overline{1,4}) \tag{32}$$

(ε is a formal small parameter). The zeroth approximation corresponds to steady-state jet flow, and, according to expressions (1), we therefore have

$$U^{(0)}(\zeta) = \zeta,\qquad F^{(0)}(\zeta) = 0,\qquad c_j^{(0)} = a_j.$$

The first approximation corresponds to some small perturbations propagating in steady-state jets.

Substituting expansion (32) into Eqs. (24)–(26) and equating the terms at the first power of the small parameter ε, we obtain equations for the first correction to the steady-state solution $c_j^{(1)}$, $F^{(1)}$, $U^{(1)}$, which will be less lengthy if, instead of the last two series (32) we use one series

$$Z(\zeta,t;\varepsilon) = Z^{(0)}(\zeta) + \varepsilon Z^{(1)}(\zeta,t) + \ldots, \tag{33}$$

and instead of formulas (24)–(26), we use Eqs. (10) and (11).

From relations (1), we obtain the zeroth approximation and its derivative

$$Z^{(0)} = \frac{1}{\pi} \sum_{j=1}^{4} \frac{h_j}{a_j} \ln\Big(1 - \frac{\zeta}{a_j}\Big), \qquad Z_\zeta^{(0)} = -\frac{1}{\pi} \sum \frac{h_j}{a_j^2} \frac{1}{1 - \zeta/a_j}. \tag{34}$$

The function $Z_\zeta^{(0)}$ and the function $R^{(0)}(\zeta) = 1/Z_\zeta^{(0)}$ have a number of important properties which are used to solve the boundary-value problems considered below.

In addition to series (32) and (33), it is also convenient to use auxiliary series for the functions included in Eqs. (10) and (11):

$$G(\zeta, t; \varepsilon) = G^{(0)}(\zeta) + \varepsilon G^{(1)}(\zeta, t) + \ldots, \qquad Q(\zeta, t; \varepsilon) = Q^{(0)}(\zeta) + \varepsilon Q^{(1)}(\zeta, t) + \ldots,$$

$$A(t; \varepsilon) = A^{(0)} + \varepsilon A^{(1)}(t) + \ldots, \qquad B(t; \varepsilon) = B^{(0)} + \varepsilon B^{(1)}(t) + \ldots.$$

We show that the steady-state solution (zeroth approximation) is indeed a solution of Eqs. (10) and (11). Passing to the limit $\varepsilon \to 0$ in (10) and (11), we obtain

$$G^{(0)} = 0, \qquad -A^{(0)}/\zeta + \overline{A^{(0)}}\, \zeta + iB^{(0)} = Q^{(0)}, \tag{35}$$

where

$$G^{(0)}(\zeta, t) = S(i\zeta\overline{\zeta}), \qquad Q^{(0)}(\zeta, t) = S\Big(\frac{\overline{\zeta}}{\zeta Z_\zeta^{(0)}}\Big). \tag{36}$$

Equations (35) are valid because, first, steady-state flows always obey condition (14) and, hence $A^{(0)} = 0$ and $B^{(0)} = 0$; second, both functions (36) are identically equal to zero. For the first function, this is obvious because on the unit circle, $\overline{\zeta} = 1/\zeta$. The vanishing of the second function follows from the following lemma.

Lemma 1. *For $|\zeta| = 1$, the following identity holds:*

$$\overline{\zeta^2 Z_\zeta^{(0)}} = -\zeta^2 Z_\zeta^{(0)}. \tag{37}$$

Proof. We multiply the derivative $Z_\zeta^{(0)}$ in (34) by ζ^2. Next, we perform complex conjugation, i.e., replace the quantities $\overline{\zeta}$ and $\overline{a}_j$ by $1/\zeta$ and $1/a_j$, respectively. As a result, on the unit circle, the following equalities hold:

$$-\zeta^2 Z_\zeta^{(0)} = \frac{1}{\pi} \sum \frac{h_j}{a_j^2} \frac{\zeta^2}{1 - \zeta/a_j}, \qquad \overline{\zeta^2 Z_\zeta^{(0)}} = \frac{1}{\pi} \sum \frac{h_j a_j}{\zeta} \frac{1}{1 - \zeta/a_j}. \tag{38}$$

Using identities (2)–(4), we prove that expressions (38) are equal. Sequentially reducing the power ζ in the first expression (38), we obtain the relation

$$-\zeta^2 Z_\zeta^{(0)} = \frac{\zeta}{\pi} \sum \frac{h_j}{a_j} \frac{\zeta/a_j - 1 + 1}{1 - \zeta/a_j} = \frac{\zeta}{\pi} \sum \frac{h_j}{a_j} \frac{1}{1 - \zeta/a_j} = \frac{1}{\pi} \sum h_j \frac{\zeta/a_j - 1 + 1}{1 - \zeta/a_j}$$

$$= \frac{1}{\pi} \sum h_j \frac{1}{1 - \zeta/a_j} = \frac{1}{\pi} \sum \frac{h_j a_j}{\zeta} \frac{\zeta/a_j - 1 + 1}{1 - \zeta/a_j} = \frac{1}{\pi} \sum \frac{h_j a_j}{\zeta} \frac{1}{1 - \zeta/a_j},$$

which coincides with the second expression in (38). Thus, lemma 1 is proved.

One more nonobvious property of the function $Z_\zeta^{(0)}$ is formulated in the following lemma.

Lemma 2. *The function $R^{(0)}(\zeta) = 1/Z_\zeta^{(0)}$ is a fourth-degree polynomial of ζ.*

Proof. We reduce the sum of the second formula (34) to a common denominator. As a result, the numerator contains the expression

$$\sum_{j=1}^{4} h_j \prod_{k \neq j} a_k^2 \Big(1 - \frac{\zeta}{a_k}\Big). \tag{39}$$

Let us prove that the third-degree polynomial (39) is a constant, i.e., the coefficients of the polynomial at all powers of ζ, except for zero, are equal to zero.

As an example, we consider the coefficient at ζ^1:

$$-a_1^2a_2^2a_3^2a_4^2\Big[\frac{1}{a_1a_4}\Big(\frac{h_1}{a_1}+\frac{h_4}{a_4}\Big)+\frac{1}{a_1a_3}\Big(\frac{h_1}{a_1}+\frac{h_3}{a_3}\Big)+\frac{1}{a_1a_2}\Big(\frac{h_1}{a_1}+\frac{h_2}{a_2}\Big)$$

$$+\frac{1}{a_2a_4}\Big(\frac{h_2}{a_2}+\frac{h_4}{a_4}\Big)+\frac{1}{a_2a_3}\Big(\frac{h_2}{a_2}+\frac{h_3}{a_3}\Big)+\frac{1}{a_3a_4}\Big(\frac{h_3}{a_3}+\frac{h_4}{a_4}\Big)\Big].$$

Using (4), we replace each two terms in parentheses by two others. As a result, we have

$$a_1^2a_2^2a_3^2a_4^2\Big[\frac{1}{a_1a_4}\Big(\frac{h_2}{a_2}+\frac{h_3}{a_3}\Big)+\frac{1}{a_1a_3}\Big(\frac{h_2}{a_2}+\frac{h_4}{a_4}\Big)+\frac{1}{a_1a_2}\Big(\frac{h_3}{a_3}+\frac{h_4}{a_4}\Big)$$

$$+\frac{1}{a_2a_4}\Big(\frac{h_1}{a_1}+\frac{h_3}{a_3}\Big)+\frac{1}{a_2a_3}\Big(\frac{h_1}{a_1}+\frac{h_4}{a_4}\Big)+\frac{1}{a_3a_4}\Big(\frac{h_1}{a_1}+\frac{h_2}{a_2}\Big)\Big].$$

After some regrouping, we obtain

$$a_1a_2a_3a_4[a_1(h_2+h_3+h_4)+a_2(h_1+h_3+h_4)+a_3(h_1+h_2+h_4)+a_4(h_1+h_2+h_3)].$$

Simplifying each sum in parentheses by using formula (2), we have

$$-a_1a_2a_3a_4(a_1h_1+a_2h_2+a_3h_3+a_4h_4). \tag{40}$$

According to (3), Eq. (40) is equal to zero. Similarly, it is proved that the coefficients at the other powers ζ are also equal to zero. Thus,

$$R^{(0)}=\frac{1}{Z_\zeta^{(0)}}=-\frac{1}{M_2}\Big(1-\frac{\zeta}{a_1}\Big)\Big(1-\frac{\zeta}{a_2}\Big)\Big(1-\frac{\zeta}{a_3}\Big)\Big(1-\frac{\zeta}{a_4}\Big),$$

where M_2 is a constant defined by formula (5). Lemma 2 is proved.

First Approximation. We derive the equations for the first approximation. Note that, in contrast to $F^{(1)}$, the first approximation of $Z^{(1)}$ is not an analytic function at the points $\zeta=a_j$. Indeed, differentiation of (18) with respect to ε shows that, at these points, the function $Z^{(1)}$ has poles

$$Z^{(1)}=\frac{1}{\pi}\sum\frac{h_j}{a_j^3}\,\frac{\zeta c_j^{(1)}}{1-\zeta/a_j}+F^{(1)}.$$

Differentiating the first and second equations in (10) with respect to ε and then passing to the limit $\varepsilon\to 0$, we obtain

$$i\zeta(Z_\zeta^{(0)}U_t^{(1)}-Z_t^{(1)})+G^{(1)}=0,$$

where

$$G^{(1)}=S(i\zeta U_\zeta^{(1)}\,\overline{U^{(0)}}\,)+S(i\zeta U_\zeta^{(0)}\,\overline{U^{(1)}}\,)=S(iU_\zeta^{(1)})+S(i\zeta\,\overline{U^{(1)}}\,). \tag{41}$$

To find Schwarz operators in (41), we expand the complex velocity in the Taylor series

$$U^{(1)}(\zeta,t)=u_0+u_1\zeta+u_2\zeta^2+\ldots.$$

The function $iU_\zeta^{(1)}$ is analytic, but for $\zeta=0$, its imaginary part is not equal to zero. Therefore, the first Schwarz operator in (41) is equal to $iU_\zeta^{(1)}-i\operatorname{Re}u_1$. The second Schwarz operator in (41) is found from the formulas

$$S(i\zeta\,\overline{U^{(1)}}\,)=-S\Big(\frac{iU^{(1)}}{\zeta}\Big)=-S\Big(i\,\frac{U^{(1)}-u_0}{\zeta}+\frac{\overline{iu_0}}{\overline{\zeta}}\Big)=-S\Big(\frac{iU^{(1)}}{\zeta}-\frac{iu_0}{\zeta}-i\,\overline{u_0}\,\zeta\Big)=-\frac{iU^{(1)}}{\zeta}+\frac{iu_0}{\zeta}+i\,\overline{u_0}\,\zeta+iC.$$

Here the real constant C is determined from the condition that the imaginary part of the operator S vanishes for $\zeta=0$:

$$C=-\lim_{\zeta\to 0}\operatorname{Re}\Big(-\frac{U^{(1)}}{\zeta}+\frac{u_0}{\zeta}+\overline{u_0}\,\zeta\Big)=\operatorname{Re}u_1.$$

Thus, we have

$$G^{(1)}=iU_\zeta^{(1)}-i(U^{(1)}-u_0)/\zeta+i\,\overline{u_0}\,\zeta.$$

Similarly, differentiating Eq. (11) with respect to ε and passing to the limit $\varepsilon \to 0$, we obtain

$$\frac{Z_t^{(1)}}{\zeta Z_\zeta^{(0)}} - \frac{A^{(1)}}{\zeta} + \overline{A^{(1)}}\,\zeta + iB^{(1)} = Q^{(1)}, \qquad Q^{(1)} = S\Big(\frac{\overline{U^{(1)}}}{\zeta Z_\zeta^{(0)}}\Big) - S\Big(\frac{Z_\zeta^{(1)}}{\zeta^2 (Z_\zeta^{(0)})^2}\Big). \tag{42}$$

In (42), an arbitrary normalization of the conformal mapping was used. If we use the normalization (14), we have $A^{(1)} = 0$ and $B^{(1)} = 0$.

Let us find the first Schwarz operator in the expression for $Q^{(1)}$. According to the conformality condition, $Z_\zeta^{(0)} \neq 0$ and, hence, the following power series exists:

$$U^{(1)}/Z_\zeta^{(0)} = b_0 + b_1\zeta + b_2\zeta^2 + b_3\zeta^3 + \ldots. \tag{43}$$

Using series (43) and equality (37) written as $\overline{\zeta Z_\zeta^{(0)}} = -\zeta^3 Z_\zeta^{(0)}$, we obtain

$$S\Big(\frac{\overline{U^{(1)}}}{\zeta Z_\zeta^{(0)}}\Big) = S\Big(\frac{U^{(1)}}{\overline{\zeta Z_\zeta^{(0)}}}\Big) = -S\Big(\frac{U^{(1)}}{\zeta^3 Z_\zeta^{(0)}}\Big) = -S\Big(\frac{U^{(1)}/Z_\zeta^{(0)} - b_0 - b_1\zeta - b_2\zeta^2}{\zeta^3} + \overline{b_0}\,\zeta^3 + \overline{b_1}\,\zeta^2 + \overline{b_2}\,\zeta\Big)$$

$$= -\frac{U^{(1)}}{\zeta^3 Z_\zeta^{(0)}} + \frac{b_0}{\zeta^3} + \frac{b_1}{\zeta^2} + \frac{b_2}{\zeta} - \overline{b_0}\,\zeta^3 - \overline{b_1}\,\zeta^2 - \overline{b_2}\,\zeta + iC_1. \tag{44}$$

Passing to the limit $\zeta \to 0$ in expression (44), we have

$$\lim_{\zeta\to 0} S\Big(\frac{\overline{U^{(1)}}}{\zeta Z_\zeta^{(0)}}\Big) = -b_3 + iC_1.$$

According to the definition of the operator S, the imaginary part of this quantity should be equal to zero; therefore, $C_1 = \operatorname{Im} b_3$. Thus, the first Schwarz operator in (42) is found.

We now find the second Schwarz operator in (42). Considering the series

$$Z_\zeta^{(1)}/(Z_\zeta^{(0)})^2 = d_0 + d_1\zeta + d_2\zeta^2 + \ldots, \tag{45}$$

we similarly find

$$S\Big(\frac{Z_\zeta^{(1)}}{\zeta^2 (Z_\zeta^{(0)})^2}\Big) = S\Big(\frac{Z_\zeta^{(1)}/(Z_\zeta^{(0)})^2 - d_0 - d_1\zeta}{\zeta^2} + \overline{d_0}\,\zeta^2 + \overline{d_1}\,\zeta\Big) = \frac{Z_\zeta^{(1)}}{\zeta^2 (Z_\zeta^{(0)})^2} - \frac{d_0}{\zeta^2} - \frac{d_1}{\zeta} + \overline{d_0}\,\zeta^2 + \overline{d_1}\,\zeta + iC_2. \tag{46}$$

For $\zeta \to 0$, the quantity (46) is equal to $d_2 + iC_2$; therefore, $C_2 = -\operatorname{Im} d_2$. From this, it follows that

$$Q^{(1)} = -\frac{U^{(1)}}{\zeta^3 Z_\zeta^{(0)}} - \frac{Z_\zeta^{(1)}}{\zeta^2 (Z_\zeta^{(0)})^2} + \frac{b_0}{\zeta^3} + \frac{b_1}{\zeta^2} + \frac{b_2}{\zeta} - \overline{b_0}\,\zeta^3 - \overline{b_1}\,\zeta^2 - \overline{b_2}\,\zeta + \frac{d_0}{\zeta^2} + \frac{d_1}{\zeta} - \overline{d_0}\,\zeta^2 - \overline{d_1}\,\zeta + i\operatorname{Im}(b_3 + d_2).$$

Denoting the coefficients at the power ζ^{j-3} by $q_j(t)$, we obtain the following system of equations for the evolution of small perturbations propagating on the steady-state solution:

$$\begin{gathered} -\zeta Z_t^{(1)} + U_\zeta^{(1)} - \frac{U^{(1)}}{\zeta} + U_t^{(1)} \zeta Z_\zeta^{(0)} = -\frac{u_0}{\zeta} - \overline{u_0}\,\zeta, \\ Z_t^{(1)} \zeta Z_\zeta^{(0)} + Z_\zeta^{(1)} + U^{(1)}\,\frac{Z_\zeta^{(0)}}{\zeta} = \zeta^2 (Z_\zeta^{(0)})^2 \Big(\frac{q_0}{\zeta^3} + \frac{q_1}{\zeta^2} + \frac{q_2}{\zeta} + q_3 - \overline{q_0}\,\zeta^3 - \overline{q_1}\,\zeta^2 - \overline{q_2}\,\zeta - \overline{q_3}\Big). \end{gathered} \tag{47}$$

The right sides of system (47), which has a nonstandard form, contain the functions of time $u_0(t)$ and $q_j(t)$ which are expressed in terms of the unknown functions $Z^{(1)}(\zeta, t)$ and $U^{(1)}(\zeta, t)$ and their derivatives for $\zeta = 0$:

$$u_0 = U^{(1)}(0,t), \qquad q_0 = b_0, \qquad q_1 = b_1 + d_0,$$
$$q_2 = b_2 + d_1 + A^{(1)}, \qquad q_3 = (b_3 + d_2 - iB^{(1)})/2.$$

More detailed expressions for $q_j(t)$ can be found by analyzing series (43) and (45).

Conclusions. Equations describing the evolution of perturbations propagating on impinging jets were derived and analyzed. Nonlinear equations were obtained which describe the evolution of jets for the case where the perturbations are not assumed small and which can be used for the numerical solution of the problem. Linearization of these equations leads to nonstandard equations which in [17] were called loaded equations because their right sides contain the solution and its derivatives taken at the coordinate origin.

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REFERENCES

1. L. M. Milne-Thomson, *Theoretical Hydrodynamics*, Dover, New York (1996).
2. G. R. Cowan, O. R. Bergman, and A. H. Holtzman, "Mechanism of bond zone wave formation in explosion-clad metals," *Metallurg. Mater. Transact.*, **2**, No. 11, 3145–3155 (1971).
3. A. A. Deribas, *Hardening Physics and Explosion Welding* [in Russian], Nauka, Novosibirsk (1980).
4. *Wave Formation During Oblique Collisions* (collected scientific papers) [in Russian], Inst. of Discrete Math. and Computer Science, Novosibirsk (2000).
5. J. L. Fox and G. W. Morgan, "On the stability of some flows of an ideal fluid with free surfaces," *Quart. Appl. Math.*, **11**, No. 4, 439–456 (1954).
6. A. Palatini, "Sulla confluenza di due vene," *Atti del R. Inst. Venetto di Sc. L. ed Arti.*, **75**, 451–463 (1916).
7. S. A. Kinelovskii, "Concept of elastic-inelastic interactions of jet flow and collision of jets of an ideal incompressible fluid," *Combust., Expl., Shock Waves*, **30**, No. 3, 331–339 (1994).
8. Yu. A. Trishin, "Asymmetric impact of a jets with an ideal noncompressible liquid," *J. Appl. Mech. Tech. Phys.*, No. 5, 663–667 (1986).
9. J. Hureau and R. Weber, "Impinging free jets of ideal fluid," *J. Fluid Mech.*, **372**, 357–374 (1998).
10. L. V. Ovsyannikov, "Plane problem of the unsteady motion of a fluid with free boundaries," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 8, Inst. of Hydrodynamics, Sib. Div., Russian Acad. of Sci., Novosibirsk (1971), pp. 22–26.
11. A. A. Dorodnitsyn, "Plane problem of the unsteady motion of a heavy fluid," in: *Proc. Int. Symp. on Applications of Function Theory in the Mechanics of Continuous Media* [in Russian], Vol. 2, Nauka, Moscow (1965), pp. 171–172.
12. E. A. Karabut, "Application of power series in time in the problem of motion of a cylindrical cavity in a fluid: 1. Finding the coefficients of power series," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 78, Inst. of Hydrodynamics, Sib. Div., Russian Acad. of Sci., Novosibirsk (1986), pp. 56–73.
13. A. I. D'yachenko, "On the dynamics of an ideal fluid with a free surface," *Dokl. Ross. Akad. Nauk.*, **376**, No. 1, 27–29 (2001).
14. V. E. Zakharov, A. I. Dyachenko, and O. A. Vasilyev, "New method for numerical simulation of a unsteady potential flow of incompressible fluid with a free surface," *Europ. J. Mech. B*, **21**, 283–291 (2002).
15. R. V. Shamin, *Computational Experiments in Modeling Surface Waves in the Ocean* [in Russian], Nauka, Moscow (2008).
16. N. S. Kozin, "Small perturbations of a jet flowing from a slot," *Prikl. Mat. Mekh.*, **36**, No. 4, 641–646 (1972).
17. E. A. Bondarev and A. F. Voevodin, "Difference method for the solution of initial-boundary-value problems for loaded differential and integro differential equations," *Differ. Uravn.*, **36**, No. 11, 1560–1562 (2000).